Stationary Flows of the Parabolic Potential Barrier in Two Dimensions

Toshiki Shimbori* and Tsunehiro Kobayashi†

*Institute of Physics, University of Tsukuba

Ibaraki 305-8571, Japan

†Department of General Education for the Hearing Impaired, Tsukuba College of Technology

Ibaraki 305-0005, Japan

Abstract

In the two-dimensional isotropic parabolic potential barrier $V(x,y) = V_0 - m\gamma^2(x^2+y^2)/2$, though it is a model of an unstable system in quantum mechanics, we can obtain the stationary states corresponding to the real energy eigenvalue V_0. Further, they are infinitely degenerate. For the first few eigenstates, we will find the stationary flows round a right angle that are expressed by the complex velocity potentials $W = \pm\gamma z^2/2$.

1 Introduction

It is well known that the two-dimensional harmonic oscillator is equivalent to the dynamical system consisting of the two independent one-dimensional harmonic oscillators—the energy eigenvalues of the two-dimensional oscillator are given by the sum of the energy eigenvalues of the one-dimensional oscillators and the eigenstates of the system are given by the product of the eigenstates of the one-dimensional oscillators. When degenerate eigenstates of the two-dimensional oscillator are superposed with a suitable weights, the new states will be the eigenstates of orbital angular momentum. These results were studied a long time ago by Dirac [1].

In the present paper we will investigate the two-dimensional parabolic potential barrier, which is a model of an unstable system in quantum mechanics, on the same lines as the two-dimensional harmonic oscillator. This model is equivalent to the dynamical system consisting of the two independent one-dimensional parabolic potential barriers. The one-dimensional potential barrier was studied by the authors [2, 3]. It is shown that the energy eigenvalues are complex numbers and the corresponding eigenfunctions are expressible in terms of the generalized functions of a Gel'fand triplet. In two dimensions the exact solutions of the eigenvalue problem of this model separate into four types. We will take the types of two of the four types in § 3.1, for which the solutions are expressed by generalized eigenfunctions belonging to complex energy eigenvalues and represented by diverging and converging flows. In these two types the solutions will also be the eigenstates of orbital angular momentum. We will study the other two of the four types in § 3.2. In these two types all the solutions are infinitely degenerate and involve the special solutions with real energy eigenvalue. Such special solutions are represented by stationary corner flows. From the hydrodynamical point of view they can be described by some kind of quantum "velocity" and complex velocity potential discussed in a previous paper [4]. It has been pointed out to us by R. Jackiw that this velocity was originally introduced half a century ago by Madelung [5]. Such a velocity is still useful in present-day high-energy physics [6]. We shall see that, for the first few solutions, the velocities reviewed in § 2 are solenoidal, so the corresponding complex velocity potentials must exist. These complex velocity potentials for the two-dimensional parabolic potential barrier describe the flows round a right angle.

2 Complex velocity potentials in quantum mechanics

In the present section we shall summarize the features of the velocity in quantum mechanics.

Define the velocity of a state $\psi(t, \boldsymbol{r})$ in non-relativistic quantum mechanics by

$$\boldsymbol{v} \equiv \frac{\boldsymbol{j}(t, \boldsymbol{r})}{|\psi(t, \boldsymbol{r})|^2}, \tag{2.1}$$

where $\boldsymbol{j}(t, \boldsymbol{r})$ is the probability current

$$\boldsymbol{j}(t, \boldsymbol{r}) \equiv \Re \left[\psi(t, \boldsymbol{r})^* \left(-i\hbar\nabla\right) \psi(t, \boldsymbol{r})\right] / \, m, \tag{2.2}$$

and m is the mass of the particle.

If all variables are separable, this velocity is in general irrotational, namely, the vorticity defined by

$$\boldsymbol{\omega} \equiv \nabla \times \boldsymbol{v} \tag{2.3}$$

vanishes. Then the velocity in irrotational flow may be described by the gradient of the velocity potential $\varPhi$,

$$\boldsymbol{v} = \nabla \varPhi. \tag{2.4}$$

We now proceed to study only the two-dimensional flow. Let us consider the velocity (2.1) which is solenoidal, namely

$$\nabla \cdot \boldsymbol{v} \equiv \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0. \tag{2.5}$$

The velocity in two-dimensional flow satisfying this solenoidal condition may be described by the rotation of the stream function $\varPsi$,

$$v_x = \frac{\partial \varPsi}{\partial y}, \;\; v_y = -\frac{\partial \varPsi}{\partial x}. \tag{2.6}$$

Further, in the two-dimensional irrotational flow, equations (2.4) and (2.6) can be combined into Cauchy-Riemann's equations between the velocity potential and the stream function. We can therefore take the complex velocity potential

$$W(z) = \varPhi(x, y) + i\varPsi(x, y), \tag{2.7}$$

which is a regular function of the complex variable $z = x + iy$. For example, the flow round the angle π/a is expressed by

$$W = Az^a, \tag{2.8}$$

A being a number. With $a = 1$, this expresses the uniform flow of the two-dimensional plane wave in quantum mechanics. There are some elementary examples of complex velocity potentials which express the two-dimensional flows in quantum mechanics [4].

The above-mentioned method will be applied to the stationary flows of the two-dimensional parabolic potential barrier in § 3.2.

3 The parabolic potential barrier in two dimensions

The Hamiltonian of the two-dimensional isotropic parabolic potential barrier is

$$\hat{H} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + V_0 - \frac{1}{2}m\gamma^2\left(x^2 + y^2\right), \tag{3.1}$$

where $V_0 \in \mathbb{R}$ is the maximum potential energy, $m > 0$ is the mass and $\gamma > 0$ is proportional to the square root of the curvature at $(x, y) = (0, 0)$.

A state is represented by a wave function $U(x, y)$ satisfying the Schrödinger equation, which now reads, with $\hat{H}$ given by (3.1),

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)U(x,y) + \left\{V_0 - \frac{1}{2}m\gamma^2\left(x^2 + y^2\right)\right\}U(x,y) = EU(x,y). \tag{3.2}$$

The energy eigenvalues of (3.2) will be the sum of the energy eigenvalues of the one-dimensional parabolic potential barrier in the x-direction and y-direction, respectively, i.e.

$$E_{n_x n_y} = E_{n_x} + E_{n_y} \tag{3.3}$$

and the eigenfunctions belonging to these energy eigenvalues will be the product of their corresponding eigenfunctions

$$U_{n_x n_y}(x, y) = u_{n_x}(x)u_{n_y}(y). \tag{3.4}$$

With the notation of preceding papers [2, 3], the energy eigenvalues of the one-dimensional parabolic potential barrier are

$$E^{\pm}_{n_q} = \frac{1}{2}V_0 \mp i\left(n_q + \frac{1}{2}\right)\hbar\gamma \quad (n_q = 0, 1, 2, \cdots) \tag{3.5}$$

and the corresponding eigenfunctions are

$$u^{\pm}_{n_q}(q) = e^{\pm i\beta^2 q^2/2}H^{\pm}_{n_q}(\beta q) \quad \left(\beta \equiv \sqrt{m\gamma/\hbar}\right), \tag{3.6}$$

where $H^{\pm}_{n_q}(\beta q)$ are the polynomials of degree n_q, and the numerical coefficients are discarded. The eigenfunctions $u^{\pm}_{n_q}$ are generalized functions in $\mathcal{S}(\mathbb{R})^{\times}$ of the following Gel'fand triplet,

$$\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset \mathcal{S}(\mathbb{R})^{\times}, \tag{3.7}$$

where $L^2(\mathbb{R})$ is a Lebesgue space and $\mathcal{S}(\mathbb{R})$ is a Schwartz space. The work [2] also shows that the index $+$ means only outward moving particles and the index $-$ means only inward moving particles.

Thus the results (3.3) and (3.4) of the two-dimensional parabolic potential barrier separate into four types:

Type 1.
$$E^{++}_{n_xn_y} = E^{+}_{n_x} + E^{+}_{n_y} = V_0 - i(n_x + n_y + 1)\hbar\gamma,$$
$$U^{++}_{n_xn_y}(x,y) = u^{+}_{n_x}(x)u^{+}_{n_y}(y) = e^{+i\beta^2(x^2+y^2)/2}H^{+}_{n_x}(\beta x)H^{+}_{n_y}(\beta y).$$

Type 2.
$$E^{+-}_{n_xn_y} = E^{+}_{n_x} + E^{-}_{n_y} = V_0 - i(n_x - n_y)\hbar\gamma,$$
$$U^{+-}_{n_xn_y}(x,y) = u^{+}_{n_x}(x)u^{-}_{n_y}(y) = e^{+i\beta^2(x^2-y^2)/2}H^{+}_{n_x}(\beta x)H^{-}_{n_y}(\beta y).$$

Type 3.
$$E^{-+}_{n_xn_y} = E^{-}_{n_x} + E^{+}_{n_y} = V_0 + i(n_x - n_y)\hbar\gamma,$$
$$U^{-+}_{n_xn_y}(x,y) = u^{-}_{n_x}(x)u^{+}_{n_y}(y) = e^{-i\beta^2(x^2-y^2)/2}H^{-}_{n_x}(\beta x)H^{+}_{n_y}(\beta y).$$

Type 4.
$$E^{--}_{n_xn_y} = E^{-}_{n_x} + E^{-}_{n_y} = V_0 + i(n_x + n_y + 1)\hbar\gamma,$$
$$U^{--}_{n_xn_y}(x,y) = u^{-}_{n_x}(x)u^{-}_{n_y}(y) = e^{-i\beta^2(x^2+y^2)/2}H^{-}_{n_x}(\beta x)H^{-}_{n_y}(\beta y).$$

These eigenfunctions $U^{++}_{n_xn_y}$, $U^{+-}_{n_xn_y}$, $U^{-+}_{n_xn_y}$, $U^{--}_{n_xn_y}$ are also generalized functions in $\mathcal{S}(\mathbb{R}^2)^\times$ of the Gel'fand triplet

$$\mathcal{S}(\mathbb{R}^2) \subset L^2(\mathbb{R}^2) \subset \mathcal{S}(\mathbb{R}^2)^\times \tag{3.8}$$

instead of (3.7). Note that the eigenfunctions of types 4 and 3 are conjugate complex functions of types 1 and 2, respectively, i.e.

$$U^{\pm\pm}_{n_xn_y}(x,y)^* = U^{\mp\mp}_{n_xn_y}(x,y)$$

and

$$U^{\pm\mp}_{n_xn_y}(x,y)^* = U^{\mp\pm}_{n_xn_y}(x,y).$$

3.1 Diverging and converging flows

Let us consider first the types 1 and 4. In this case the energy eigenvalues $E^{\pm\pm}_{n_xn_y}$ are always complex numbers and the time factors corresponding to them are

$$e^{-iE^{\pm\pm}_{n_xn_y}t/\hbar} = e^{-iV_0t/\hbar}e^{\mp(n_x+n_y+1)\gamma t}.$$

Thus the solutions of type 1 are well defined when $t > 0$, and those of type 4 are well defined when $t < 0$, according to the time boundary condition that time factors of an unstable system are square integrable [2]. Also, $U^{++}_{n_xn_y}(x,y)$ represent particles moving outward from the center as in fig. 1, and $U^{--}_{n_xn_y}(x,y)$ represent particles moving inward to the center as in fig. 2. Thus we shall call these types *diverging* and *converging flows*, respectively. Note that a time reversal occurs resulting in the interchange of the diverging and converging flows.

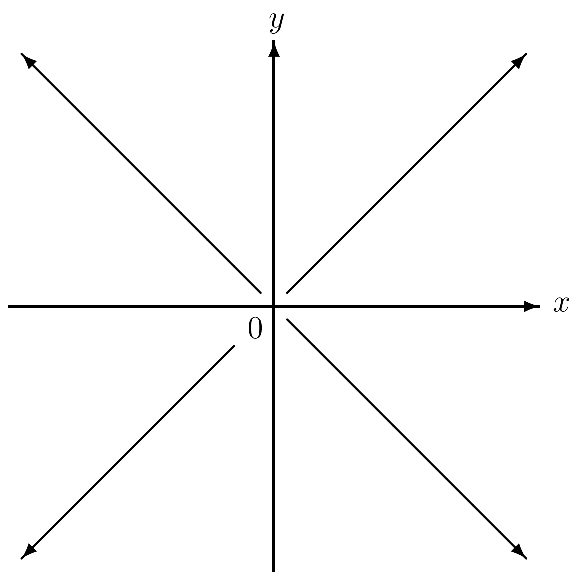

Fig. 1: Diverging flows.

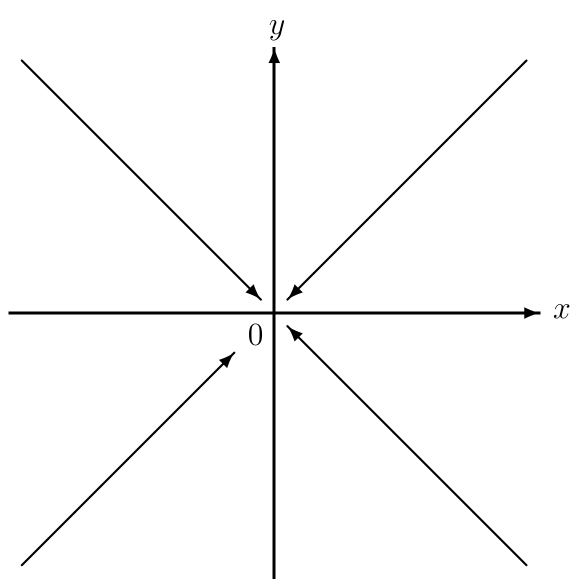

Fig. 2: Converging flows.

For $n_x = n_y = 0$, we get the energy eigenvalue

$$E_{00}^{\pm\pm} = V_0 \mp i\hbar\gamma \tag{3.9}$$

and only one eigenfunction

$$U_{00}^{\pm\pm}(x, y) = e^{\pm i\beta^2(x^2+y^2)/2}, \tag{3.10}$$

respectively. For $n_x + n_y = 1$, namely $n_x = 1, n_y = 0$ and $n_x = 0, n_y = 1$, we get

$$E_{10}^{\pm\pm} = E_{01}^{\pm\pm} = V_0 \mp 2i\hbar\gamma \tag{3.11}$$

and two eigenfunctions

$$\left.\begin{aligned} U_{10}^{\pm\pm}(x, y) &= 2\beta x e^{\pm i\beta^2(x^2+y^2)/2}, \\ U_{01}^{\pm\pm}(x, y) &= 2\beta y e^{\pm i\beta^2(x^2+y^2)/2}. \end{aligned}\right\} \tag{3.12}$$

There is twofold degenerate state of types 1 and 4 with $n_x + n_y = 1$. For $n_x + n_y = 2$, namely $n_x = 2, n_y = 0$; $n_x = 1, n_y = 1$; $n_x = 0, n_y = 2$, we get

$$E_{20}^{\pm\pm} = E_{11}^{\pm\pm} = E_{02}^{\pm\pm} = V_0 \mp 3i\hbar\gamma \tag{3.13}$$

and three eigenfunctions

$$\left.\begin{aligned} U_{20}^{\pm\pm}(x, y) &= (4\beta^2 x^2 \mp 2i) e^{\pm i\beta^2(x^2+y^2)/2}, \\ U_{11}^{\pm\pm}(x, y) &= 4\beta^2 xy e^{\pm i\beta^2(x^2+y^2)/2}, \\ U_{02}^{\pm\pm}(x, y) &= (4\beta^2 y^2 \mp 2i) e^{\pm i\beta^2(x^2+y^2)/2}. \end{aligned}\right\} \tag{3.14}$$

There is threefold degenerate state of types 1 and 4 with $n_x + n_y = 2$. Generally, there is $(n+1)$-fold degenerate state of types 1 and 4 with $n_x + n_y = n$. This result is just the same degree of degeneracy as the two-dimensional harmonic oscillator.

For the further discussion of the state of types 1 and 4, we now pass from the Cartesian coordinates x, y to the two-dimensional polar coordinates r, φ by means of the equations

$$\left.\begin{aligned} x &= r\cos\varphi, \\ y &= r\sin\varphi. \end{aligned}\right\} \tag{3.15}$$

If in the new coordinates we superpose above-mentioned eigenstates with suitable weights, the result will be the eigenstates of orbital angular momentum $\hat{L}$ defined by

$$\hat{L} = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) = -i\hbar\frac{\partial}{\partial\varphi}. \tag{3.16}$$

For $n_x = n_y = 0$, the eigenfunction (3.10) will be

$$U_{00}^{\pm\pm}(r, \varphi) = e^{\pm i\beta^2 r^2/2}. \tag{3.17}$$

Thus $U_{00}^{\pm\pm}(r,\varphi)$ is independent of φ and has zero orbital angular momentum. For $n_x+n_y=1$, a linear combination of the eigenfunctions (3.12) give

$$\left.\begin{aligned} U_{10}^{\pm\pm}(r,\varphi)+iU_{01}^{\pm\pm}(r,\varphi) &= 2\beta r e^{\pm i\beta^2r^2/2}e^{i\varphi},\\ U_{10}^{\pm\pm}(r,\varphi)-iU_{01}^{\pm\pm}(r,\varphi) &= 2\beta r e^{\pm i\beta^2r^2/2}e^{-i\varphi}.\end{aligned}\right\} \tag{3.18}$$

These states are eigenstates of $\hat{L}$ with eigenvalues $\hbar$ and $-\hbar$, respectively. For $n_x+n_y=2$, (3.14) give

$$\left.\begin{aligned} U_{20}^{\pm\pm}(r,\varphi)+2iU_{11}^{\pm\pm}(r,\varphi)-U_{02}^{\pm\pm}(r,\varphi) &= 4\beta^2r^2e^{\pm i\beta^2r^2/2}e^{2i\varphi},\\ U_{20}^{\pm\pm}(r,\varphi)+U_{02}^{\pm\pm}(r,\varphi) &= 4(\beta^2r^2\mp i)e^{\pm i\beta^2r^2/2},\\ U_{20}^{\pm\pm}(r,\varphi)-2iU_{11}^{\pm\pm}(r,\varphi)-U_{02}^{\pm\pm}(r,\varphi) &= 4\beta^2r^2e^{\pm i\beta^2r^2/2}e^{-2i\varphi}.\end{aligned}\right\} \tag{3.19}$$

These states are also eigenstates of $\hat{L}$ with eigenvalues $2\hbar$, 0, and $-2\hbar$. There is a similar procedure for large values of n_x+n_y.

3.2 Corner flows

Let us now study the types 2 and 3. In this case the energy eigenvalues $E_{n_xn_y}^{\pm\mp}$ are also in general complex numbers, but with the striking difference that all the eigenstates belonging to each energy eigenvalue are infinitely degenerate. The corresponding time factors are

$$e^{-iE_{n_xn_y}^{\pm\mp}t/\hbar}=e^{-iV_0t/\hbar}e^{\mp(n_x-n_y)\gamma t}.$$

Thus the solutions of type 2 are well defined when $t>0$, and those of type 3 are well defined when $t<0$, for the case of $n_x>n_y$, and vice versa. Also, $U_{n_xn_y}^{+-}(x,y)$ represent particles which, coming from the y-direction, round the center and, go off to the x-direction as in fig. 3, and $U_{n_xn_y}^{-+}(x,y)$ represent particles which, coming from the x-direction, round the center and, go off to the y-direction as in fig. 4. Note that a time reversal occurs resulting in the interchange of these corner flows.

Stationary flows

The above time factors now show that for the case of $n_x=n_y$, there are *stationary flows*. For $n_x=n_y=n=0,1,2,\cdots$, the energy eigenvalues associated with stationary flows are the same real number:

$$E_{nn}^{\pm\mp}=V_0. \tag{3.20}$$

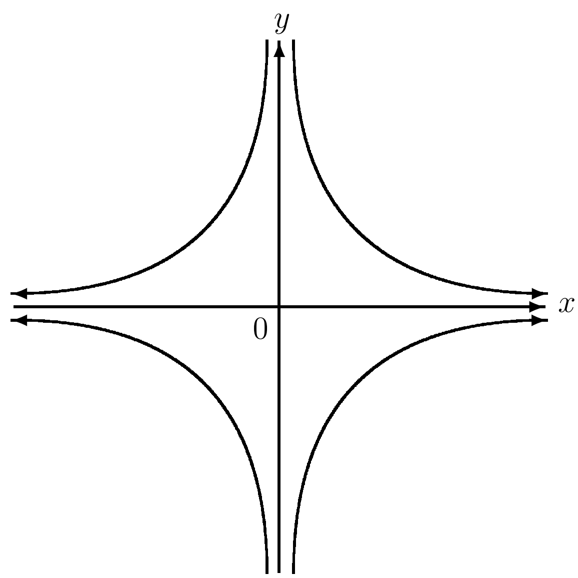

FIG. 3: Corner flows moving from the y-direction to the x-direction.

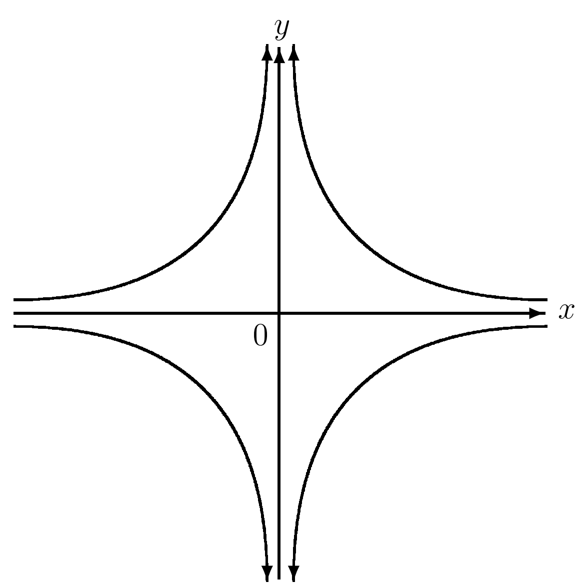

FIG. 4: Corner flows moving from the x-direction to the y-direction.

The first few infinitely degenerate eigenfunctions belonging to this energy eigenvalue (3.20) are

$$\left.\begin{aligned} U_{00}^{\pm\mp}(x,y) &= e^{\pm i\beta^2(x^2-y^2)/2}, \\ U_{11}^{\pm\mp}(x,y) &= 4\beta^2 xy e^{\pm i\beta^2(x^2-y^2)/2}, \\ U_{22}^{\pm\mp}(x,y) &= 4\left[4\beta^4 x^2 y^2 + 1 \pm 2i\beta^2(x^2-y^2)\right] e^{\pm i\beta^2(x^2-y^2)/2}, \\ &\ldots. \end{aligned}\right\} \tag{3.21}$$

For the further study of the stationary flows of the two-dimensional parabolic potential barrier with the Hamiltonian (3.1), it is convenient to make a transformation to the rectangular hyperbolic coordinates u, v, given by

$$\left.\begin{aligned} u &= x^2 - y^2, \\ v &= 2xy. \end{aligned}\right\} \tag{3.22}$$

The eigenfunctions (3.21) will become in the new representation

$$\left.\begin{aligned} U_{00}^{\pm\mp}(u,v) &= e^{\pm i\beta^2 u/2}, \\ U_{11}^{\pm\mp}(u,v) &= 2\beta^2 v e^{\pm i\beta^2 u/2}, \\ U_{22}^{\pm\mp}(u,v) &= 4\left(\beta^4 v^2 + 1 \pm 2i\beta^2 u\right) e^{\pm i\beta^2 u/2}, \\ &\ldots. \end{aligned}\right\} \tag{3.23}$$

The factors $e^{\pm i\beta^2 u/2}$ occurred in (3.23) describe plane waves in the uv-plane, i.e. the motion of the wave $e^{i\beta^2 u/2}$ is in the direction specified by fig. 3 and that of the wave $e^{-i\beta^2 u/2}$ is in the direction specified by fig. 4. These eigenfunctions substituted in (2.2) give the rectangular hyperbolic coordinates $j_{nnu}^{\pm\mp}$, $j_{nnv}^{\pm\mp}$ of $\boldsymbol{j}_{nn}^{\pm\mp}$, which are the probability currents of the states $U_{nn}^{\pm\mp}$. They give

$$\begin{aligned} &j_{00u}^{\pm\mp}(u,v) = \pm\gamma h_u/2, \;\; j_{00v}^{\pm\mp}(u,v) = 0, \\ &j_{11u}^{\pm\mp}(u,v) = \pm 2\gamma\beta^4 v^2 h_u, \;\; j_{11v}^{\pm\mp}(u,v) = 0, \\ &j_{22u}^{\pm\mp}(u,v) = \pm 8\gamma\left\{(\beta^4 v^2 + 5)(\beta^4 v^2 + 1) + 4\beta^4 u^2\right\} h_u, \\ &j_{22v}^{\pm\mp}(u,v) = \mp 64\gamma\beta^4 uv h_v, \\ &\qquad\ldots, \end{aligned}$$

where the scale factors $h_u = h_v = 2\sqrt[4]{u^2+v^2}$. Thus $\boldsymbol{j}_{nn}^{\pm\mp}$ can never depend on the time t. We see from this result the suitability of the term "stationary flows".

The flows round a right angle

The above probability currents now show that for the case of $n = 0$ and 1, there are stationary flows which move along the hyperbolas (each line with v constant). To get an

understanding of the physical features of this flows it is better to work with the velocity defined by (2.1). For $n = 0$ and 1, the velocities give the same result

$$v_u^{\pm\mp} = \pm\frac{1}{2}\gamma h_u, \;\; v_v^{\pm\mp} = 0.$$

Taking the rotation of them, the vorticity (2.3) becomes

$$\omega^{\pm\mp} \equiv h_u h_v \left[\frac{\partial}{\partial u}\left(\frac{v_v^{\pm\mp}}{h_v}\right) - \frac{\partial}{\partial v}\left(\frac{v_u^{\pm\mp}}{h_u}\right)\right] = 0.$$

These equations will hold generally in quantum mechanics [4], and therefore the velocity potentials defined by (2.4) must exist. If we transform to rectangular hyperbolic coordinates u, v, equations (2.4) become

$$v_u = h_u \frac{\partial \varPhi}{\partial u}, \;\; v_v = h_v \frac{\partial \varPhi}{\partial v}, \tag{3.24}$$

and the velocity potentials for $n = 0$ and 1 are thus

$$\varPhi^{\pm\mp} = \pm\frac{1}{2}\gamma u. \tag{3.25}$$

Note that they are proportional to the phase factors of (3.23). Further, the divergence (2.5) gives

$$\nabla \cdot \boldsymbol{v}^{\pm\mp} \equiv h_u h_v \left[\frac{\partial}{\partial u}\left(\frac{v_u^{\pm\mp}}{h_v}\right) + \frac{\partial}{\partial v}\left(\frac{v_v^{\pm\mp}}{h_u}\right)\right] = 0.$$

Thus the velocities are solenoidal for $n = 0$ and 1, so that we can obtain the stream functions defined by (2.6). The equations (2.6) are also expressed, as in equations (3.24),

$$v_u = h_v \frac{\partial \varPsi}{\partial v}, \;\; v_v = -h_u \frac{\partial \varPsi}{\partial u}, \tag{3.26}$$

and the stream functions for $n = 0$ and 1 are thus

$$\varPsi^{\pm\mp} = \pm\frac{1}{2}\gamma v. \tag{3.27}$$

For the states represented by the first and second of equations (3.21) or (3.23), the complex velocity potential (2.7) gives, from (3.25) and (3.27)

$$\begin{aligned} W^{\pm\mp} &= \pm\frac{1}{2}\gamma u \pm \frac{i}{2}\gamma v \\ &= \pm\frac{1}{2}\gamma z^2, \end{aligned} \tag{3.28}$$

since $z^2 = u + iv$. Equations (3.28) are of the form (2.8) with $a = 2$, and they show that, *for* $n = 0$ *and* 1, *the complex velocity potentials of the two-dimensional parabolic potential barrier express the flows round a right angle*. One could work out in terms of Cartesian coordinates and one would be led to the same conclusion.

4 Discussion

We have obtained the exact solutions of the two-dimensional parabolic potential barrier. One class of the solutions is diverging and converging flows of § 3.1. These solutions are always complex energy eigenvalues and generalized eigenfunctions, which mean that the diverging and converging flows are not stationary. These generalized eigenfunctions can be obtained from the eigenfunctions of the two-dimensional harmonic oscillator by the analytical continuation, in the same way as the one-dimensional parabolic potential barrier [2]. Again, they can be superposed to give the eigenstates of orbital angular momentum. For these solutions, however, the method mentioned in § 2 was not applicable. As an example we try to calculate the divergence (2.5) for the states (3.10) or (3.17). The result is

$$\nabla \cdot \boldsymbol{v}_{00}^{\pm\pm} = \pm 2\gamma \neq 0.$$

Now the $\pm$ sign shows that U_{00}^{++} is connected with the diverging flow and U_{00}^{--} is connected with the converging one. Thus the velocities of diverging and converging flows cannot be solenoidal. Therefore the stream functions and the complex velocity potentials do not exist. This result still holds for large values of $n_x + n_y$.

The other class of the solutions is corner flows of § 3.2. All the solutions are infinitely degenerate and involve the stationary flows with real energy eigenvalue. It should be noted that there are no stationary flows in the one-dimensional or three-dimensional *isotropic* parabolic potential barrier. For $n = 0$ and 1 in the stationary flows, we have found the flows round a right angle that are expressed by the complex velocity potentials (3.28). But for $n \geqslant 2$, the complex velocity potentials do not exist, because the imaginary parts of the polynomials $H_{n_q}^{\pm}(\beta q)$ in (3.6) cause the stream lines to depart from hyperbolas. One may, however, find a new flow as the result of a kind of superposition of the infinitely degenerate states.

One would expect to be able to get a more direct solution of the eigenvalue problem of the Hamiltonian (3.1) by working all the time in the two-dimensional polar coordinates, instead of working in the Cartesian coordinates and transforming at the end to the two-dimensional polar coordinates, as was done in § 3.1. But under suitable boundary conditions in the two-dimensional polar coordinates, one would obtain only diverging and converging flows of § 3.1, i.e. the lack of corner flows of § 3.2. It is also pointed out that one can get the only diverging and converging flows from the analytical continuation of the solutions of the two-dimensional harmonic oscillator. These facts mean that *the choice of coordinate systems is quite important in the eigenvalue problem of the unstable system in non-relativistic quantum mechanics, since coordinate systems impose a restriction on the symmetry of boundary conditions.* The source of the conclusion lies in the existence of a very large class of solutions for the unstable system.

It is rather surprising that such a stable idea as stationary flows should appear in the parabolic potential barrier in this way. Actually we have shown in another paper [7] that the dynamical system composed of several of these potential barriers forms the quasi-stable semiclassical system.

References

[1] P. A. M. Dirac, *The Principles of Quantum Mechanics* (Clarendon Press, 1930).

[2] T. Shimbori and T. Kobayashi, *Complex Eigenvalues of the Parabolic Potential Barrier and Gel'fand Triplet*, math-ph/9910009, to appear in Nuovo Cim. **B**.

[3] T. Shimbori, *Operator Methods of the Parabolic Potential Barrier*, quant-ph/9912073.

[4] T. Shimbori and T. Kobayashi, *"Velocities" in Quantum Mechanics*, quant-ph/0004086.

[5] E. Madelung, Z. Phys. **40** (1926) 322.

[6] For example, D. Bazeia and R. Jackiw, Ann. Phys. **270** (1998) 246.

[7] T. Kobayashi and T. Shimbori, *Statistical Mechanics for States with Complex Eigenvalues and Quasi-stable Semiclassical Systems*, cond-mat/0005237.